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SPURIOUS ENERGY DISSIPATION/GENERATION IN STIFFNESS RECOVERY MODELS FOR ELASTIC DEGRADATION AND DAMAGE

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Abstract—A number of constitutive models have been proposed in recent years for elastic degradation and damage, many of which include procedures for the recovery of stiffness upon closure of tensile microcracks. Most of these recovery procedures are based on the decomposition of stress or strain into positive (tensile) and negative (compressive) components, which are incorporated in the elastic formulation taking recourse to fourth-order positive and negative projection operators. Due to the non-dissipative nature of microcrack closure–reopening for a certain fixed state of degradation, the recovery formulation should possess a well-defined energy potential along the line of hyperelasticity, which conserves energy upon closed-loop load histories. This condition seems to have escaped the rapidly expanding literature on damage mechanics, i.e. closure formulations have not been verified in this regards. In the paper, the (lack of) energy conservation is examined in terms of the spurious dissipation rate, which is developed for a relatively general class of recovery models. They include the positive–negative projection operators and the bimodular formulations with different stiffnesses for tension and compression. It is shown that under proportional loading in strain or stress, all these formulations are energy conservative. Under non-proportional loading, however, they are only conservative in conjunction with isotropic degradation, and they exhibit spurious dissipation–generation when anisotropic degradation is considered and the load history involves rotation of principal directions. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Constitutive models representing the degradation of stiffness have proliferated in the last two decades, starting from the family of smeared crack models to the recent developments in continuum damage formulations based on micromechanics. A unifying theory for elastic degradation and damage based on a loading surface was recently presented (Carol *et al.*, 1994), as well as the combination of elastic degradation and plasticity into a single multi-dissipative framework (Rizzi *et al.*, 1995). A particular aspect of elastic-degrading models, however, still represents a serious challenge and remains unresolved. For many materials (rock, concrete, ceramics, composites, etc.), the reduction of stiffness in tension is associated with the development of distributed microcracking. The intuitive idea that upon load reversal microcracks can close and the initial stiffness can be recovered, is supported by experimental evidence at the macroscopic scale (Reinhardt and Cornelissen, 1984), and has persuaded a number of authors to develop constitutive procedures which take into account the microcrack closure–reopening (MCR) effects (Ortiz, 1985; Mazars and Pijaudier-Cabot, 1989; Ju, 1989; La Borderie *et al.*, 1990; Chaboche, 1990; Berthaud *et al.*, 1990; Hansen, 1993; Faria and Oliver, 1993).

Procedures for MCR in recent literature are mainly based on concepts such as positive and negative decomposition of the stress and strain and the fourth-order $\mathbf{P}^+$ and $\mathbf{P}^-$ projection operators, which were first introduced by Ortiz (1985), and then used by others (Ju, 1989; Hansen, 1993). These are referred to in this paper as $\mathbf{P}^+/\mathbf{P}^-$ MCR procedures.

The basic idea of materials with different moduli in tension and compression is, however, considerably older. Elementary procedures along this line may be found in the literature on bimodulus materials, mainly devoted to composites with fibers stiffer than the matrix (see for instance the proceedings volume on the subject edited by Bert, 1979), no-tension materials (Zienkiewicz *et al.*, 1968; Alonso and Carol, 1985) for fractured rock masses, and smeared cracking models (Rashid, 1968; Darwin and Pecknold, 1976; Willam *et al.*, 1987) for materials with low tensile strength such as concrete and rock. Some authors have also used the $\mathbf{E}^+/\mathbf{E}^-$ procedures with different elastic moduli in tension and compression for the description of MCR effects (Mazars and Pijaudier-Cabot, 1989).

Among the general requirements that material models with stiffness degradation and recovery must satisfy (Carol *et al.*, 1994), this paper focuses on the conservation of energy specifically for the representation of MCR effects. The nature of these effects and the restrictions that must apply for their representation may be isolated by considering a constant or fixed state of microcracking, and load histories that cause these microcracks to open, close and reopen without further crack extension or propagation. These are realistic conditions for low-intensity loading when stresses or strains exhibit rotating principal axes and/or when the principal values change sign. In damage models based on loading functions, these conditions correspond to stress of strain histories which do not reach the current loading surface. Under those conditions and for non-proportional loading, MCR effects lead in general to non-linear stress–strain behavior. The absence of damage propagation, however, suggests the requirements of energy conservation upon closed load cycles, with a well-defined energy potential $w = w[\varepsilon]$ (in our notation square brackets enclose arguments of a function) along the line of hyperelasticity or Green elasticity (Malvern, 1969). Some authors did recently consider alternative or additional conditions which need to be satisfied by the MCR formulation, such as continuity of stresses across positive–negative domain boundaries (Chaboche, 1993) similar to the idea previously developed for bimodulus materials (Tabbador, 1979), or convexity of the corresponding energy potential (Chaboche, 1990; Pijaudier-Cabot *et al.*, 1994). However, the existence of an energy potential for MCR formulations is of a fundamental nature and has not been studied in depth in the literature. Once the potential does exist, one can discuss continuity of its derivatives (stresses) and/or convexity.

In the traditional *thermodynamic approach* of internal variables, the formulation of damage normally starts from a thermodynamic potential as a function of the strain (or stress) and of damage variables (i.e. the Helmholtz free energy $\Phi = \Phi[\varepsilon, \bar{\mathscr{D}}_*]$, where the asterisk $_*$ denotes any number of indices: 0 = scalar, 1 = vector, 2 = tensor. Consequently, the stresses and other variables are the partial derivatives of that potential (Ortiz, 1985; Mazars and Pijaudier-Cabot, 1989; Ju, 1989; La Borderie *et al.*, 1990; Chaboche, 1990; Berthaud *et al.*, 1990; Hansen, 1993; Faria and Oliver, 1993). In principle, this approach should ensure satisfaction of the energy-conservation requirement. However, this does not seem to be the case for many recovery formulations found in the literature, because the strict thermodynamic derivation is carried out in terms of the total strain or stress tensor, and MCR effects are added after the fact by introducing positive and negative projection tensors in an arbitrary way, i.e. without verifying whether the new expressions satisfy the condition $\boldsymbol{\sigma} = \partial\phi/\partial\boldsymbol{\varepsilon}$. The main difficulty for this verification stems from the fact that the new potential is a function of the positive and negative strain or stress components as independent quantities, while the stresses must be obtained as the derivative with respect to the total strain tensor; this implies two intermediate derivatives which were often overlooked, which is acceptable only if all principal directions are of the same sign, or if the principal directions remain fixed. As a consequence, only in a few cases restricted to isotropic degradation were MCR effects taken into account with a full verification of energy conservation (e.g. see Faria and Oliver, 1993).

Due to this basic deficiency, it seems appropriate to propose a methodology to verify whether any given MCR formulation does exhibit an energy potential. In this paper, such a methodology is proposed and developed through the concept of spurious dissipation rate, which must vanish if the formulation is energy conservative. This condition is developed for a relatively general class of elastic-degrading and damage formulations in which unloading always leads to the origin (no plastic strains), and for which MCR effects are introduced through active stiffness or compliance tensors, which are functions of the current strain or stress state. The paper is structured in the following way: the general concepts of energy potential and spurious dissipation are discussed and formulated in Section 2. In Section 3, $\mathbf{P}^+/\mathbf{P}^-$ models are considered. The general expression of spurious dissipation is studied for various existing projection operators and for a new one proposed in this paper. Their performance for loading histories involving rotation of principal strains is discussed for general isotropic degradation, and for a particular case of anisotropic degradation. In Section 4, the same exercise is repeated for bimodular $\mathbf{E}^+/\mathbf{E}^-$ formulations. In Section 5, a specific example of anisotropic damage model with a popular MCR procedure is shown to exhibit significant dissipation–generation of energy when subjected to a simple load history with a 90° rotation of the principal strains. Finally, the main conclusions and some remarks are presented in Section 6.

2. ENERGY CONSIDERATIONS AND SPURIOUS DISSIPATION

2.1. *Nonlinear elastic material*

The conditions for a non-linear elastic material not to exhibit energy dissipation upon closed load histories are well established in the theory of classical Green elasticity or hyperelasticity (Malvern, 1969), and can be summarized as follows: (i) there exists an elastic energy potential w function of the strain or strain state; and (ii) upon any variation of stress or strain, the increment of the energy potential equals the external work supply. In the range of small strain, without consideration of thermal effects, and assuming that unloading always leads to the origin, these conditions can be written as:

$$w = w[\boldsymbol{\varepsilon}] \quad \text{and} \quad \dot{w} = \sigma_{ij}\dot{\varepsilon}_{ij} \quad \text{or} \quad \sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}, \qquad (1\text{a, b, c})$$

where repeated indices imply summation unless otherwise indicated. If the conditions (1b, c) are satisfied, the energy function can be written as

$$w = \int_0^{\boldsymbol{\varepsilon}} \sigma_{ij}\,\mathrm{d}\varepsilon_{ij}, \qquad (2)$$

where the lower limit "0" means a reference state with zero energy potential (normally the origin) and the integral is path-independent.

For the practical application of those concepts to the present study, oriented to the verification of existing models defined as $\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{\varepsilon}]$, it is convenient to change the approach and start with the definition of a scalar function w defined as the integral (2) *following a proportional strain path from the origin*, i.e.

$$w[\boldsymbol{\varepsilon}] = \int_{\mu=0}^{\mu=1} \sigma_{ij}[\mu\boldsymbol{\varepsilon}]\varepsilon_{ij}\,\mathrm{d}\mu. \qquad (3)$$

Since, for a general stress–strain law, w defined in eqn (3) is not guaranteed to be an energy potential that satisfies eqns (1b) and (1c), the *spurious dissipation* can be introduced as the difference

$$\dot{d}^{\mathrm{sp}} = \sigma_{ij}\dot{\varepsilon}_{ij} - \dot{w}, \qquad (4)$$

that, after development of $\dot{w}$, can be also written as

$$\dot{d}^{\rm sp} = \sigma_{ij}\dot{\varepsilon}_{ij} - \frac{\partial w}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} = \left(\sigma_{ij} - \frac{\partial w}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij}. \tag{5}$$

With these definitions, the verification that w is a true energy potential reduces to the verification that $\dot{d}^{\rm sp} \equiv 0$, since then (1b, c) are automatically satisfied.

2.2. *Elastic-degrading material*

The preceding concepts can be generalized for an elastic-degrading solid for which the stiffness is reduced progressively and unloading always leads to the origin. In this case, the energy potential depends not only on the state of strain, but also on the current state of degradation represented by some (internal) damage variables $\bar{\mathscr{D}}_*$ (the asterisk represents the desired number of indices: 0 = scalar, 1 = vector, 2 = tensor; repetition implies summation over all indices), and one can write

$$w = w[\boldsymbol{\varepsilon}, \bar{\mathscr{D}}_*]\,; \quad \dot{w} = \left.\frac{\partial w}{\partial \varepsilon_{ij}}\right|_{\mathscr{D}} \dot{\varepsilon}_{ij} + \left.\frac{\partial w}{\partial \bar{\mathscr{D}}_*}\right|_{\varepsilon} \dot{\bar{\mathscr{D}}}_*, \tag{6a, b}$$

where the variables $\boldsymbol{\varepsilon}$ and $\bar{\mathscr{D}}$ in eqn (6b) are assumed to be constant for the partial derivatives (this notation is used throughout the paper). Condition (6b) can be rewritten in the form of an energy balance equation (first principle):

$$\dot{w} = \sigma_{ij}\dot{\varepsilon}_{ij} - \dot{d}^{\rm d}, \tag{7}$$

where the stress tensor σ_{ij} is defined as

$$\sigma_{ij} = \left.\frac{\partial w}{\partial \varepsilon_{ij}}\right|_{\mathscr{D}}, \tag{8}$$

and $\dot{d}^{\rm d}$ is the *degrading dissipation*, which must remain positive to ensure irreversibility of damage (second principle) and can be expressed in terms of the *generalized or thermodynamic force* $-\bar{\mathscr{Y}}_*$ conjugate to $\bar{\mathscr{D}}_*$.

$$\dot{d}^{\rm d} = \left.\frac{\partial w}{\partial \bar{\mathscr{D}}_*}\right|_{\varepsilon} \dot{\bar{\mathscr{D}}}_* = -\bar{\mathscr{Y}}_*\dot{\bar{\mathscr{D}}}_* \geqslant 0 \quad \text{with} \quad -\bar{\mathscr{Y}}_* = \left.\frac{\partial w}{\partial \bar{\mathscr{D}}_*}\right|_{\varepsilon}. \tag{9a, b}$$

For the purpose of verification of degradation models of the form $\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{\varepsilon}, \bar{\mathscr{D}}_*]$, it is appropriate to start defining the scalar function w in the following way:

$$w[\boldsymbol{\varepsilon}, \bar{\mathscr{D}}_*] = \int_{\mu=0}^{\mu=1} \sigma_{ij}[\mu\boldsymbol{\varepsilon}, \bar{\mathscr{D}}_*]\varepsilon_{ij}\,{\rm d}\mu. \tag{10}$$

Since, for a general stress–strain law, w in eqn (10) is not necessarily an energy potential that satisfies eqns (7) and (8), the rate of spurious dissipation can be introduced as

$$\dot{d}^{\rm sp} = \sigma_{ij}\dot{\varepsilon}_{ij} - d^{\rm d} - \dot{w}. \tag{11}$$

After replacing eqns (6b) and (9a), the spurious dissipation may be expressed as

$$\dot{d}^{\rm sp} = \sigma_{ij}\dot{\varepsilon}_{ij} - \left.\frac{\partial w}{\partial \varepsilon_{ij}}\right|_{\mathscr{D}} \dot{\varepsilon}_{ij} = \left(\sigma_{ij} - \left.\frac{\partial w}{\partial \varepsilon_{ij}}\right|_{\mathscr{D}}\right)\dot{\varepsilon}_{ij} \tag{12}$$

Consequently, verification that w given by eqn (10) is a well-defined energy potential reduces checking that $\dot{d}^{sp} \equiv 0$, since this automatically implies that eqns (7) and (8) are satisfied. Note that eqn (12) is equivalent to eqn (5) for non-linear elastic materials. For elastic-degrading models, this condition must be satisfied in addition to the positiveness of degrading dissipation, eqn (9a). However, both conditions are entirely independent and uncoupled, and normally different parts of the constitutive formulation are responsible for each of them; evolution laws for damage based on a loading surface and a flow rule determine $\dot{d}^{d}$, while unloading–reloading behavior for a constant state of degradation is solely responsible for $\dot{d}^{sp}$. Focusing on the purpose of the present discussion, only the unloading–reloading aspects of the formulation will be considered in the following sections.

2.3. $\dot{d}^{sp}$ *for a class of elastic-degrading models*

Following the notation of Carol *et al.* (1994), let $\mathbf{E}^0$ be the initial stiffness assumed isotropic and $\bar{\mathscr{D}}_*$ the damage variables of unspecified nature (scalar, vector, tensor...) characterizing the state of degradation. MCR effects are included through some non-constant *active stiffness* tensor defined as a function of the state of degradation and current strain values $\mathbf{E}^{ac} = \mathbf{E}^{ac}[\bar{\mathscr{D}}_*, \boldsymbol{\varepsilon}]$, in such a way that for no damage $\mathbf{E}^{ac}[\bar{\mathscr{D}}_* = 0, \boldsymbol{\varepsilon}] \equiv \mathbf{E}^0$.

For a certain *fixed* state of damage, which is the case under scrutiny, the active stiffness can be considered a function of strain only, i.e. $\mathbf{E}^{ac} = \mathbf{E}^{ac}[\boldsymbol{\varepsilon}]$. For strain states which activate all microcracks (normally tensile in all directions), the active stiffness should be equal to the intrinsic secant stiffness $\mathbf{E}$ that corresponds to the fixed state of damage; for strains states under which all microcracks are closed (normally compression in all directions), the active stiffness should be equal to the initial stiffness $\mathbf{E}^0$; and for intermediate states the active stiffness should provide smooth transition between the two. In a similar fashion, one can consider the dual stress-based approach with an initial isotropic compliance $\mathbf{C}^0$, an intrinsic secant compliance (assuming all microcracks open) $\mathbf{C}$ and an active secant compliance $\mathbf{C}^{ac} = \mathbf{C}^{ac}[\boldsymbol{\sigma}]$ providing smooth transition between both limiting situations

The basic equations for both strain- and stress-based approaches are

$$\sigma_{ij} = E^{ac}_{ijkl}[\boldsymbol{\varepsilon}]\varepsilon_{kl} \quad \text{or} \quad \varepsilon_{ij} = C^{ac}_{ijkl}[\boldsymbol{\sigma}]\sigma_{kl}. \tag{13a, b}$$

This is satisfied by all expressions of $\mathbf{E}^{ac}$ and $\mathbf{C}^{ac}$ based on positive/negative tensorial decomposition that will be considered later in the paper. The additional assumption is made that *upon proportional loading, the active stiffness* (*or compliance*) *remains constant*, i.e. for any positive scalar $k > 0$,

$$E^{ac}_{ijkl}[k\boldsymbol{\varepsilon}] = E^{ac}_{ijkl}[\boldsymbol{\varepsilon}] \quad \text{or} \quad C^{ac}_{ijkl}[k\boldsymbol{\sigma}] = C^{ac}_{ijkl}[\boldsymbol{\sigma}]. \tag{14a, b}$$

With these assumptions, the integral in eqn (10) yields

$$w = \tfrac{1}{2}\varepsilon_{ij}E^{ac}_{ijkl}\varepsilon_{kl} \quad \text{or} \quad w = \tfrac{1}{2}\sigma_{ij}C^{ac}_{ijkl}\sigma_{kl}, \tag{15a, b}$$

independently of the stress- or strain-based approach used. $\mathbf{E}^{ac}$ and $\mathbf{C}^{ac}$ are the inverse moduli of each other, and therefore eqns (15a) and (15b) are fully equivalent. Equation (15a) can be differentiated (assuming a fixed damage state) and introduced into eqn (4) to obtain the rate of spurious dissipation:

$$\dot{d}^{sp} = -\tfrac{1}{2}\varepsilon_{ij}\dot{E}^{ac}_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \dot{d}^{sp} = \tfrac{1}{2}\sigma_{ij}\dot{C}^{ac}_{ijkl}\sigma_{kl}. \tag{16a, b}$$

Thereby eqn (16b) may be obtained by differentiating $\mathbf{E}^{ac}:\mathbf{C}^{ac} = \mathbf{I}^4$ (fourth-order identity tensor) and replacing $\dot{\mathbf{E}}^{ac} = -\mathbf{E}^{ac}:\dot{\mathbf{C}}^{ac}:\mathbf{E}^{ac}$ into eqn (16a).

3. $\mathbf{P}^+$ AND $\mathbf{P}^-$-BASED FORMULATIONS

3.1. *Spectral decomposition of stress or strain; introduction of* $\mathbf{P}^+$ *and* $\mathbf{P}^-$

The separation of stresses (or strains) into some positive and negative components is founded on the spectral representation of the second-order tensor

$$\sigma_{ij} = \sum_{\alpha=1}^{3} \sigma^{(\alpha)} n_i^{(\alpha)} n_j^{(\alpha)} \quad \text{or} \quad \varepsilon_{ij} = \sum_{\alpha=1}^{3} \varepsilon^{(\alpha)} \bar{n}_i^{(\alpha)} \bar{n}_j^{(\alpha)}, \tag{17a, b}$$

where the indices between the parentheses are not summed (this convention is used throughout the paper). $\sigma^{(\alpha)}$ designates the αth principal stress, $n_i^{(\alpha)}$ the corresponding normalized principal direction, and $\varepsilon^{(\alpha)}$ and $\bar{n}_i^{(\alpha)}$ are the same for strain. From eqn (17a) the definition of positive and negative parts of stresses are

$$\sigma_{ij}^{+} = \sum_{\alpha=1}^{3} \langle \sigma^{(\alpha)} \rangle n_i^{(\alpha)} n_j^{(\alpha)}; \quad \sigma_{ij}^{-} = \sigma_{ij} - \sigma_{ij}^{+} = \sum_{k=1}^{3} \langle -\sigma^{(\alpha)} \rangle n_i^{(\alpha)} n_j^{(\alpha)}, \tag{18a, b}$$

where "$\langle . \rangle$" denotes the McAuley brackets with the usual definition $\langle x \rangle = (x + |x|)/2$. Therefore, the sum in eqn (18a) holds only for the positive eigenvalues and in eqn (18b) for the negative ones. Analogous expressions can be obtained for the positive and negative strain measures:

$$\varepsilon_{ij}^{+} = \sum_{\alpha=1}^{3} \langle \varepsilon^{(\alpha)} \rangle \bar{n}_i^{(\alpha)} \bar{n}_j^{(\alpha)}; \quad \varepsilon_{ij}^{-} = \varepsilon_{ij} - \varepsilon_{ij}^{+} = \sum_{k=1}^{3} \langle -\varepsilon^{(\alpha)} \rangle \bar{n}_i^{(\alpha)} \bar{n}_j^{(\alpha)}. \tag{19a, b}$$

The projection operators for stresses $\mathbf{P}^+$ and $\mathbf{P}^-$ as introduced by Ortiz (1985) must extract the positive and negative components as defined by eqns (18a, b) or (19a, b)

$$\sigma_{ij}^{+} = P_{ijkl}^{+} \sigma_{kl}; \quad \sigma_{ij}^{-} = P_{ijkl}^{-} \sigma_{kl} \quad \text{or} \quad \varepsilon_{ij}^{+} = \bar{P}_{ijkl}^{+} \varepsilon_{kl}; \quad \varepsilon_{ij}^{-} = \bar{P}_{ijkl}^{-} \varepsilon_{kl}. \tag{20a, b, c, d}$$

In general, $\mathbf{P}^+$ and $\mathbf{P}^-$ (or $\bar{\mathbf{P}}^+$ and $\bar{\mathbf{P}}^-$) will depend on the state of stress (or strain) and their expressions are not unique, as shown in the following subsections. Since $\boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^- = \boldsymbol{\sigma}$ (or $\boldsymbol{\varepsilon}^+ + \boldsymbol{\varepsilon}^- = \boldsymbol{\varepsilon}$), one has the requirement that $\mathbf{P}^+ + \mathbf{P}^- = \mathbf{I}^4$ (or $\bar{\mathbf{P}}^+ + \bar{\mathbf{P}}^- = \mathbf{I}^4$).

3.2. *Active secant stiffness*

Most expressions for active stiffness in eqn (13a) are based on the following expression proposed by Ortiz (1985) for the active compliance, see eqn (13b):

$$C_{ijkl}^{\mathrm{ac}} = C_{ijkl}^{0} + \Delta C_{ijkl}^{\mathrm{ac}}; \quad \Delta C_{ijkl}^{\mathrm{ac}} = P_{ijpq}^{+} \Delta C_{pqrs}^{+} P_{rskl}^{+} + P_{ijpq}^{-} \Delta C_{pqrs}^{-} P_{rskl}^{-}, \tag{21a, b}$$

where ΔC_{pqrs}^{+} and ΔC_{pqrs}^{-} would be the intrinsic increase of compliance with all microcracks active in tension or compression respectively. The dual strain-based counterparts of eqn (21a, b) are

$$E_{ijkl}^{\mathrm{ac}} = E_{ijkl}^{0} - \Delta E_{ijkl}^{\mathrm{ac}}; \quad \Delta E_{ijkl}^{\mathrm{ac}} = \bar{P}_{ijpq}^{+} \Delta E_{pqrs}^{+} \bar{P}_{rskl}^{+} + \bar{P}_{ijpq}^{-} \Delta E_{pqrs}^{-} \bar{P}_{rskl}^{-}. \tag{22a, b}$$

Following Ju (1989) and Hansen (1993), the strain-based expression (22a) with only the positive term on the right-hand side of eqn (22b) will be considered in the following. Analogous results are obtained by considering the dual stress-based formulation or including the negative term in the equations. The case considered corresponds to $\Delta E_{pqrs}^{-} = 0$ and $\Delta E_{pqrs}^{+} = \Delta E_{pqrs} = E_{pqrs} - E_{pqrs}^{0}$, that reduces eqn (22a, b) into

$$E^{ac}_{ijkl} = E^{0}_{ijkl} - \bar{P}^{+}_{ijpq}\Delta E_{pqrs}\bar{P}^{+}_{rskl}. \tag{23}$$

3.3. *Spurious dissipation*

Differentiation of eqn (23) for constant ΔE_{pqrs} yields

$$\dot{E}^{ac}_{ijkl} = -\dot{\bar{P}}^{+}_{ijpq}\Delta E_{pqrs}\bar{P}^{+}_{rskl} - \bar{P}^{+}_{ijpq}\Delta E_{pqrs}\dot{\bar{P}}^{+}_{rskl}. \tag{24}$$

Introducing eqn (24) into eqn (16a) and assuming major symmetry for ΔE_{pqrs}, one obtains the spurious dissipation rate

$$\dot{d}^{sp} = \varepsilon_{ij}\bar{P}^{+}_{ijpq}\Delta E_{pqrs}\dot{\bar{P}}^{+}_{rskl}\varepsilon_{kl}. \tag{25}$$

To proceed further, particular expressions must be introduced for the decrease of intrinsic stiffness and for the projection operators. The specific expressions for the degradation of stiffness are addressed first.

3.3.1. *General isotropic degradation.* The most common type of stiffness degradation used in the literature is the isotropic degradation. In this type of formulation, the decrease of intrinsic stiffness ΔE_{pqrs} always remains isotropic, and therefore so does the resulting secant stiffness E_{pqrs}. A general expression for this is

$$\Delta E_{pqrs} = \Delta\lambda\delta_{pq}\delta_{rs} + \Delta\mu(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}). \tag{26}$$

Introducing this into eqn (25), one obtains

$$\dot{d}^{sp} = \Delta\lambda\varepsilon_{ij}\bar{P}^{+}_{ijpp}\dot{\bar{P}}^{+}_{rrkl}\varepsilon_{kl} + \Delta\mu\varepsilon_{ij}(\bar{P}^{+}_{ijpq}\dot{\bar{P}}^{+}_{pqkl} + \bar{P}^{+}_{ijpq}\dot{\bar{P}}^{+}_{pqkl})\varepsilon_{kl}. \tag{27}$$

3.3.2. *A specific form of anisotropic degradation.* In the case of anisotropic degradation, a general expression of the reduction of intrinsic stiffness is not available. Instead, a specific formulation will be considered. This will suffice as a counterexample, to show that for this type of models spurious dissipation may indeed take place. The decrease of intrinsic stiffness has been assumed of the form

$$\Delta E_{pqrs} = \Delta E d_p d_q d_r d_s \tag{28}$$

where ΔE represents the reduction of initial directional stiffness, for a certain direction $\mathbf{d}$ ($d_i d_i = 1$, unit vector) in which the main reduction of stiffness occurs. This version of anisotropic stiffness degradation is the dual counterpart to the increase of compliance proposed by Ortiz (1985), when that model is subjected to uniaxial tension. Further insight on the physical meaning of eqn (28) may be obtained by considering the directional stiffness between normal stress and strain components on a plane of orientation $\mathbf{n}$ ($n_i n_i = 1$, unit vector), given by $E^{(n)} = E_{pqrs}n_p n_q n_r n_s$. The same concept applied to the decrease of intrinsic stiffness yields in this case $\Delta E^{(n)} = \Delta E d_p d_q d_r d_s n_p n_q n_r n_s = \Delta E \cos^4\theta$, where $\cos\theta = n_i d_i$, which means that θ is the angle between $\mathbf{n}$ and $\mathbf{d}$. Therefore, the reduction of directional stiffness has a maximum and is equal to ΔE in the orientation $\mathbf{d}$, and it decreases progressively (with the fourth power of the cosine) to a zero value of orientations perpendicular to $\mathbf{d}$.

If eqn (28) is now introduced into eqn (25), the following expression is obtained for the spurious dissipation:

$$\dot{d}^{\mathrm{sp}} = \Delta E \varepsilon_{ij} \bar{P}^{+}_{ijpq} d_p d_q d_r d_s \dot{\bar{P}}^{+}_{rskl} \varepsilon_{kl}. \tag{29}$$

3.4. $\dot{\mathrm{d}}^{sp}$ *for the projection operator by Ortiz* (1985)

In his original proposal, Ortiz adopted the following projection operators:

$$P^{+}_{ijkl} = \sum_{\alpha=1}^{3} \mathscr{H}[\sigma^{(\alpha)}] n_i^{(\alpha)} n_j^{(\alpha)} n_k^{(\alpha)} n_l^{(\alpha)}; \quad P^{-}_{ijkl} = I_{ijkl} - P^{+}_{ijkl} \tag{30a, b}$$

where $\mathscr{H}[.]$ denotes the Heaviside function (that is null for negative and equal to 1 for positive values of the argument). Therefore, eqn (30a) is equivalent to summation of all positive eigenvalues. I_{ijkl} is the fourth-order identity tensor, which may refer to the symmetric one $I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$ or to the non-symmetric one $I_{ijkl} = \delta_{ik}\delta_{jl}$. Complete analogous definitions are possible for $\bar{\mathbf{P}}^{+}$ and $\bar{\mathbf{P}}^{-}$ related to the strain tensor:

$$\bar{P}^{+}_{ijkl} = \sum_{\alpha=1}^{3} \mathscr{H}[\varepsilon^{\alpha}] \bar{n}_i^{(\alpha)} \bar{n}_j^{(\alpha)} \bar{n}_k^{(\alpha)} \bar{n}_l^{(\alpha)}; \quad \bar{P}^{-}_{ijkl} = I_{ijkl} - \bar{P}^{+}_{ijkl}. \tag{31a, b}$$

After these definitions, the following three combinations of positive/negative principal strains may be distinguished with their corresponding expressions for $\bar{P}^{+}$

(a) $\varepsilon^{(1)} > 0, \varepsilon^{(2)} \leqslant 0, \varepsilon^{(3)} \leqslant 0 \rightarrow \bar{P}^{+}_{ijkl} = \bar{n}_i^{(1)} \bar{n}_j^{(1)} \bar{n}_k^{(1)} \bar{n}_l^{(1)}$ (32a)

(b) $\varepsilon^{(1)} > 0, \varepsilon^{(2)} > 0, \varepsilon^{(3)} \leqslant 0 \rightarrow \bar{P}^{+}_{ijkl} = \bar{n}_i^{(1)} \bar{n}_j^{(1)} \bar{n}_k^{(1)} \bar{n}_l^{(1)} + \bar{n}_i^{(2)} \bar{n}_j^{(2)} \bar{n}_k^{(2)} \bar{n}_l^{(2)}$ (32b)

(c) $\varepsilon^{(1)} > 0, \varepsilon^{(2)} > 0, \varepsilon^{(3)} > 0 \rightarrow \bar{P}^{+}_{ijkl} = \bar{n}_i^{(1)} \bar{n}_j^{(1)} \bar{n}_k^{(1)} \bar{n}_l^{(1)} + \bar{n}_i^{(2)} \bar{n}_j^{(2)} \bar{n}_k^{(2)} \bar{n}_l^{(2)} + \bar{n}_i^{(3)} \bar{n}_j^{(3)} \bar{n}_k^{(3)} \bar{n}_l^{(3)}$. (32c)

This definition satisfies the basic equation (20c), but two arguments speak against it: (i) as indicated by Ju (1989), when all principal strains are positive, one would expect that $\bar{\mathbf{P}}^{+} = \mathbf{I}^4$, but this does not happen with expression (32c), and (ii) $\bar{\mathbf{P}}^{+}$ for one positive eigenvalue (32a) has a completely different structure from $\bar{\mathbf{P}}^{-}$ for one negative obtained with eqn (31b). Some of these aspects are improved in the alternative definitions of the projection operators considered in the next subsections, although this does not seem to affect the performance of the formulation with respect to spurious dissipation as shown later on.

In order to investigate $\dot{d}^{\mathrm{sp}}$, two cases are considered: with only one principal strain positive eqn (32a), and with two, eqn (32b). Here it is assumed that when all principal strains are positive, one takes directly eqn (22a) with $\Delta\mathbf{E}^{\mathrm{ac}} = \Delta\mathbf{E}$ instead of eqns (23) and (32c). In the first case, $\bar{\mathbf{P}}^{+}$ is given by eqn (32a). Its differentiation yields

$$\dot{\bar{P}}^{+}_{rskl} = \dot{\bar{n}}_r^{(1)} \bar{n}_s^{(1)} \bar{n}_k^{(1)} \bar{n}_l^{(1)} + \bar{n}_r^{(1)} \dot{\bar{n}}_s^{(1)} \bar{n}_k^{(1)} \bar{n}_l^{(1)} + \bar{n}_r^{(1)} \bar{n}_s^{(1)} \dot{\bar{n}}_k^{(1)} \bar{n}_l^{(1)} + \bar{n}_r^{(1)} \bar{n}_s^{(1)} \bar{n}_k^{(1)} \dot{\bar{n}}_l^{(1)}. \tag{33}$$

This and eqn (32a) can be introduced into eqn (25) to obtain the corresponding $\dot{d}^{\mathrm{sp}}$ for unspecified (but fully symmetric) stiffness degradation ΔE_{pqrs}. Using the fact that $\bar{n}_i^{(\alpha)}$ are the unit vectors with directions of principal strains and therefore $\varepsilon_{ij} \bar{n}_i^{(\alpha)} \bar{n}_j^{(\alpha)} = \varepsilon^{(\alpha)}$ and $\varepsilon_{ij} \bar{n}_i^{(\alpha)} \dot{\bar{n}}_j^{(\alpha)} = 0$, one obtains

$$\dot{d}^{\mathrm{sp}} = 2\varepsilon^{(1)^2} \Delta E_{pqrs} \bar{n}_p^{(1)} \bar{n}_q^{(1)} \bar{n}_r^{(1)} \dot{\bar{n}}_s^{(1)}. \tag{34}$$

In the second case, $\bar{\mathbf{P}}^{+}$ is given by eqn (32b). Differentiation yields

$$\dot{P}^{+}_{rskl} = \dot{\bar{n}}^{(1)}_r \bar{n}^{(1)}_s \bar{n}^{(1)}_k \bar{n}^{(1)}_l + \bar{n}^{(1)}_r \dot{\bar{n}}^{(1)}_s \bar{n}^{(1)}_k \bar{n}^{(1)}_l + \bar{n}^{(1)}_r \bar{n}^{(1)}_s \dot{\bar{n}}^{(1)}_k \bar{n}^{(1)}_l + \bar{n}^{(1)}_r \bar{n}^{(1)}_s \bar{n}^{(1)}_k \dot{\bar{n}}^{(1)}_l$$

$$+ \dot{\bar{n}}^{(2)}_r \bar{n}^{(2)}_s \bar{n}^{(2)}_k \bar{n}^{(2)}_l + \bar{n}^{(2)}_r \dot{\bar{n}}^{(2)}_s \bar{n}^{(2)}_k \bar{n}^{(2)}_l + \bar{n}^{(2)}_r \bar{n}^{(2)}_s \dot{\bar{n}}^{(2)}_k \bar{n}^{(2)}_l + \bar{n}^{(2)}_r \bar{n}^{(2)}_s \bar{n}^{(2)}_k \dot{\bar{n}}^{(2)}_l. \quad (35)$$

Substitution of this and eqn (32b) into eqn (25) leads to the following expression of the corresponding $\dot{d}^{sp}$, for an unspecified stiffness degradation ΔE_{pqrs} which possesses all major and minor symmetries:

$$\dot{d}^{sp} = 2\Delta E_{pqrs}(\varepsilon^{(1)^2}\bar{n}^{(1)}_p \bar{n}^{(1)}_q \bar{n}^{(1)}_r \dot{\bar{n}}^{(1)}_s + \varepsilon^{(2)^2}\bar{n}^{(2)}_p \bar{n}^{(2)}_q \bar{n}^{(2)}_r \dot{\bar{n}}^{(2)}_s + \varepsilon^{(1)}\varepsilon^{(2)}\bar{n}^{(1)}_p \bar{n}^{(1)}_q \bar{n}^{(2)}_r \dot{\bar{n}}^{(2)}_s$$

$$+ \varepsilon^{(1)}\varepsilon^{(2)}\bar{n}^{(2)}_p \bar{n}^{(2)}_q \bar{n}^{(1)}_r \dot{\bar{n}}^{(1)}_s). \quad (36)$$

3.4.1. *Isotropic degradation.* The expression of spurious dissipation $\dot{d}^{sp}$ for the projection operator by Ortiz leads, in the case of one single tensile principal strain and isotropic degradation, introducing eqn (26) into eqn (34), or eqns (32a) and (33) into eqn (27), to

$$\dot{d}^{sp} = 2\varepsilon^{(1)^2}(\Delta\lambda\delta_{pq}\delta_{rs} + \Delta\mu(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}))\bar{n}^{(1)}_p \bar{n}^{(1)}_q \bar{n}^{(1)}_r \dot{\bar{n}}^{(1)}_s = 0. \quad (37)$$

In this case the spurious dissipation is zero. This result reflects the fact that the products of the type $\Delta E_{pqrs}\bar{n}^{(1)}_p \bar{n}^{(1)}_q \bar{n}^{(1)}_r \dot{\bar{n}}^{(1)}_s$ vanish for isotropic ΔE_{pqrs}. This can be verified by developing the products in eqn (37) and taking into account that $\bar{n}^{(\alpha)}_i \bar{n}^{(\beta)}_i = \delta_{\alpha\beta}$ and $\bar{n}^{(\alpha)}_i \dot{\bar{n}}^{(\alpha)}_i = 0$ because they are an othornormal base.

A similar result that $\dot{d}^{sp} = 0$ is obtained when two principal directions in tension are considered, and eqn (26) is introduced in eqn (36), or eqns (32b) and (35) in eqn (27). This can be seen from eqn (36), where the additional products are of the type ΔE_{pqrs} $\bar{n}^{(1)}_p \bar{n}^{(1)}_q \bar{n}^{(2)}_r \dot{\bar{n}}^{(2)}_s$ that also vanish for isotropic ΔE_{pqrs}.

3.4.2. *Anisotropic degradation.* The spurious dissipation rate $\dot{d}^{sp}$ for one single tensile principal strain and the type of anisotropic degradation considered in Section 3.3.2, is obtained by introducing eqn (28) into eqn (34), or eqns (32a) and (33) into eqn (29). This leads to

$$\dot{d}^{sp} = 2\varepsilon^{(1)^2}\Delta E d_p d_q d_r d_s \bar{n}^{(1)}_r \bar{n}^{(1)}_s \bar{n}^{(1)}_k \dot{\bar{n}}^{(1)}_l = -2\varepsilon^{(1)^2}\Delta E\dot{\theta}\cos\theta\sin\theta, \quad (38)$$

where θ is the angle between $\mathbf{d}$ and $\bar{\mathbf{n}}^{(1)}$ and therefore $d_i\bar{n}^{(1)}_i = \cos\theta$, and since $\mathbf{d}$ is constant, $d_i\dot{\bar{n}}^{(1)}_i = -\dot{\theta}\sin\theta$. In contrast to the isotropic case, now the spurious dissipation is not zero when the principal direction of tension $\bar{\mathbf{n}}^{(1)}$ rotates with respect to the principal direction of degradation $\mathbf{d}$ and therefore $\dot{\theta} \neq 0$.

If two directions of tensile strain are considered, $\dot{d}^{sp}$ is obtained by introducing eqn (28) into eqn (36), or eqns (32b) and (35) into eqn (29). This leads to an expression similar to eqn (38) but with three more terms of the same type involving the second principal strain $\varepsilon^{(2)}$ and a second angle ϕ between $\bar{\mathbf{n}}^{(2)}$ and $\mathbf{d}$. As before, spurious dissipation takes place whenever any of the positive principal strains rotate and therefore either $\dot{\theta}$ or $\dot{\phi}$ (or both) are non-zero.

3.5. $\dot{d}^{sp}$ *for the projection operators by Simó and Ju* (1987)

Simó and Ju (1987) proposed an alternative definition of the projection operators, although it was not used for MCR effects until later (Ju, 1989; Hansen, 1993). Here, Hansen's version (where some unnecessary terms of Ju's formulation were eliminated) is considered. The expressions are:

$$P^+_{ijkl} = \left(\sum_{\alpha=1}^{3} \mathcal{H}[\sigma^{(\alpha)}]\bar{n}_i^{(\alpha)}\bar{n}_k^{(\alpha)}\right)\left(\sum_{\alpha=1}^{3} \mathcal{H}[\sigma^{(\alpha)}]\bar{n}_j^{(\alpha)}\bar{n}_l^{(\alpha)}\right); \quad P^-_{ijkl} = I_{ijkl} - P^+_{ijkl}. \qquad (39a, b)$$

From this definition, expressions for three cases may be distinguished:

$$\text{(a)} \quad \varepsilon^{(1)} > 0, \varepsilon^{(2)} \leqslant 0, \varepsilon^{(3)} \leqslant 0 \rightarrow \bar{P}^+_{ijkl} = \bar{n}_i^{(1)}\bar{n}_j^{(1)}\bar{n}_k^{(1)}\bar{n}_l^{(1)} \qquad (40a)$$

$$\text{(b)} \quad \varepsilon^{(1)} > 0, \varepsilon^{(2)} > 0, \varepsilon^{(3)} \leqslant 0 \rightarrow \bar{P}^+_{ijkl} = \bar{n}_i^{(1)}\bar{n}_j^{(1)}\bar{n}_k^{(1)}\bar{n}_l^{(1)} + \bar{n}_i^{(2)}\bar{n}_j^{(2)}\bar{n}_k^{(2)}\bar{n}_l^{(2)} + n_i^{(1)}\bar{n}_j^{(2)}\bar{n}_k^{(1)}\bar{n}_l^{(2)} + \bar{n}_i^{(2)}\bar{n}_j^{(1)}\bar{n}_k^{(2)}\bar{n}_l^{(1)} \qquad (40b)$$

$$\text{(c)} \quad \varepsilon^{(1)} > 0, \varepsilon^{(2)} > 0, \varepsilon^{(3)} > 0 \rightarrow \bar{P}^+_{ijkl} = \delta_{ik}\delta_{jl} \qquad (40c)$$

where the last result, eqn (40c), is obtained with the help of relation

$$\bar{n}_i^{(1)}\bar{n}_k^{(1)} + \bar{n}_i^{(2)}\bar{n}_k^{(2)} + \bar{n}_i^{(3)}\bar{n}_k^{(3)} = \delta_{ik} \qquad (41)$$

which holds because the three principal directions constitute an orthonormal base. With this definition, one may observe that the same operators as those proposed by Ortiz are obtained for a single tensile eigenvalue, but new expressions appear for two and three tensile eigenvalues. In particular, the latter exhibits the feature that $\bar{\mathbf{P}}^+ = \mathbf{I}^4$ (non-symmetric). However, one still has expressions with different structure for $\bar{\mathbf{P}}^+$ and $\bar{\mathbf{P}}^-$, and Simo and Ju's operators for two positive eigenvalues (40b) no longer exhibit minor symmetries (i is not interchangeable with j, and k with l). This is not important when the pairs i, j and k, l are contracted with indices of a tensor that possesses the symmetry, but matters when that is not the case, as for instance in eqn (20), with the possibility of losing symmetry of $\boldsymbol{\varepsilon}^+$. A symmetrized version of eqn (40b) may be considered to avoid those difficulties:

$$\bar{P}^+_{ijkl} = \bar{n}_i^{(1)}\bar{n}_j^{(1)}\bar{n}_k^{(1)}\bar{n}_l^{(1)} + \bar{n}_i^{(2)}\bar{n}_j^{(2)}\bar{n}_k^{(2)}\bar{n}_l^{(2)} + \tfrac{1}{2}(\bar{n}_i^{(1)}\bar{n}_j^{(2)}\bar{n}_k^{(1)}\bar{n}_l^{(2)} + \bar{n}_i^{(2)}\bar{n}_j^{(1)}\bar{n}_k^{(2)}\bar{n}_l^{(1)} + \bar{n}_i^{(1)}\bar{n}_j^{(2)}\bar{n}_k^{(2)}\bar{n}_l^{(1)} + \bar{n}_i^{(2)}\bar{n}_j^{(1)}\bar{n}_k^{(1)}\bar{n}_l^{(2)}). \qquad (42)$$

In order to investigate $\dot{d}^{sp}$, two cases are again considered: one with only one principal strain positive, and the other one with two being positive. In the first case, this operator exhibits the same expression, eqn (40a), as before, eqn (32a), and therefore its differentiation and the spurious dissipation rate $\dot{d}^{sp}$ for unspecified ΔE_{ijkl} have the same expressions as eqns (33) and (34) in the previous section.

In the second case, $\bar{\mathbf{P}}^+$ is given by eqn (40b) or by the symmetrized version of it, eqn (42). Differentiation of the original expression (40b) yields

$$\begin{aligned}\dot{\bar{P}}^+_{rskl} = {} & \dot{\bar{n}}_r^{(1)}\bar{n}_s^{(1)}\bar{n}_k^{(1)}\bar{n}_l^{(1)} + \bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)}\bar{n}_k^{(1)}\bar{n}_l^{(1)} + \bar{n}_r^{(1)}\bar{n}_s^{(1)}\dot{\bar{n}}_k^{(1)}\bar{n}_l^{(1)} + \bar{n}_r^{(1)}\bar{n}_s^{(1)}\bar{n}_k^{(1)}\dot{\bar{n}}_l^{(1)} \\ & + \dot{\bar{n}}_r^{(2)}\bar{n}_s^{(2)}\bar{n}_k^{(2)}\bar{n}_l^{(2)} + \bar{n}_r^{(2)}\dot{\bar{n}}_s^{(2)}\bar{n}_k^{(2)}\bar{n}_l^{(2)} + \bar{n}_r^{(2)}\bar{n}_s^{(2)}\dot{\bar{n}}_k^{(2)}\bar{n}_l^{(2)} + \bar{n}_r^{(2)}\bar{n}_s^{(2)}\bar{n}_k^{(2)}\dot{\bar{n}}_l^{(2)} \\ & + \dot{\bar{n}}_r^{(1)}\bar{n}_s^{(2)}\bar{n}_k^{(1)}\bar{n}_l^{(2)} + \bar{n}_r^{(1)}\dot{\bar{n}}_s^{(2)}\bar{n}_k^{(1)}\bar{n}_l^{(2)} + \bar{n}_r^{(1)}\bar{n}_s^{(2)}\dot{\bar{n}}_k^{(1)}\bar{n}_l^{(2)} + \bar{n}_r^{(1)}\bar{n}_s^{(2)}\bar{n}_k^{(1)}\dot{\bar{n}}_l^{(2)} \\ & + \dot{\bar{n}}_r^{(2)}\bar{n}_s^{(1)}\bar{n}_k^{(2)}\bar{n}_l^{(1)} + \bar{n}_r^{(2)}\dot{\bar{n}}_s^{(1)}\bar{n}_k^{(2)}\bar{n}_l^{(1)} + \bar{n}_r^{(2)}\bar{n}_s^{(1)}\dot{\bar{n}}_k^{(2)}\bar{n}_l^{(1)} + \bar{n}_r^{(2)}\bar{n}_s^{(1)}\bar{n}_k^{(2)}\dot{\bar{n}}_l^{(1)}. \end{aligned} \qquad (43)$$

Substitution of eqns (40b) and (43) into eqn (25) and some algebraic manipulation, leads to the corresponding $\dot{d}^{sp}$ for unspecified (but fully symmetric) stiffness degradation ΔE_{pqrs}. Using the symmetrized version (43) instead of eqn (42) does not change the expression of $\dot{d}^{sp}$, which is given by

$$\dot{d}^{sp} = 2\Delta E_{pqrs}(\varepsilon^{(1)^2}\bar{n}_p^{(1)}\bar{n}_q^{(1)}\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)} + \varepsilon^{(2)^2}\bar{n}_p^{(2)}\bar{n}_q^{(2)}\bar{n}_r^{(2)}\dot{\bar{n}}_s^{(2)} + \varepsilon^{(1)}\varepsilon^{(2)}\bar{n}_p^{(1)}\bar{n}_q^{(1)}\bar{n}_r^{(2)}\dot{\bar{n}}_s^{(2)}$$

$$+\varepsilon^{(1)}\varepsilon^{(2)}\bar{n}_p^{(2)}\bar{n}_q^{(2)}\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)} + \varepsilon^{(1)}\varepsilon_{ij}(\bar{n}_i^{(1)}\dot{\bar{n}}_j^{(2)} + \dot{\bar{n}}_i^{(1)}\bar{n}_j^{(2)})\bar{n}_p^{(1)}\bar{n}_q^{(1)}\bar{n}_r^{(1)}\bar{n}_s^{(2)}$$

$$+\varepsilon^{(2)}\varepsilon_{ij}(\bar{n}_i^{(1)}\dot{\bar{n}}_j^{(2)} + \dot{\bar{n}}_i^{(1)}\bar{n}_j^{(2)})\bar{n}_p^{(2)}\bar{n}_q^{(2)}\bar{n}_r^{(2)}\bar{n}_s^{(1)}). \quad (44)$$

3.5.1. *Isotropic degradation.* The expression of $\dot{d}^{sp}$ for Simo and Ju's operator with one single tensile principal strain and isotropic degradation is the same as for Ortiz's eqn (37), and therefore the spurious dissipation is always zero in this case.

If two principal directions in tension are considered, eqn (26) must be substituted in eqn (44), or eqns (40b) and (43) in eqn (27). The resulting expression for $\dot{d}^{sp}$ includes two new terms of the form $\Delta E_{pqrs}\bar{n}_p^{(1)}\bar{n}_q^{(1)}\bar{n}_r^{(1)}\bar{n}_s^{(2)}$ that also vanish for isotropic ΔE_{pqrs}, and therefore the same result is obtained that the spurious dissipation is zero.

3.5.2. *Anisotropic degradation.* The expression of $\dot{d}^{sp}$ for Simo and Ju's operator with anisotropic degradation and one single positive principal strain is the same as in Section 3.4.2. When two principal strains are positive, $\dot{d}^{sp}$ can be obtained by substituting eqn (28) into eqn (44) [or eqns (40b) and (43) into eqn (29)]. Comparison of $\dot{d}^{sp}$ for unspecified degradation, eqn (44), with its previous counterpart, eqn (36), shows two additional terms that, after substitution of eqn (28), contain factors of the type $\cos^3\theta\cos\phi$ and $\cos^3\phi\cos\theta$. These terms are non-zero in general, and they accumulate onto the already non-zero value of $\dot{d}^{sp}$ calculated in Section 3.4.2. Consequently, spurious dissipation may indeed take place in this case whenever θ or ϕ (or both) are non-constant, i.e. when the principal directions of strain rotate.

3.6. $\dot{d}^{sp}$ *for a new definition of the projection operators*

Consider the following new proposal of $\bar{\mathbf{P}}^+$ and $\bar{\mathbf{P}}^-$:

$$\bar{P}^+_{ijkl} = \sum_{\alpha=1}^{3} \mathscr{H}[\varepsilon^{(\alpha)}]\tfrac{1}{2}(\bar{n}_i^{(\alpha)}\bar{n}_k^{(\alpha)}\delta_{jl} + \delta_{ik}\bar{n}_j^{(\alpha)}\bar{n}_l^{(\alpha)})\,; \quad \bar{P}^-_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \bar{P}^+_{ijkl}. \quad (45a,b)$$

After this definition, the three cases depending on the signs of the principal strains and the corresponding expressions of $\bar{\mathbf{P}}^+$ are

$$(a)\ \varepsilon^{(1)} > 0, \varepsilon^{(2)} \leqslant 0, \varepsilon^{(3)} \leqslant 0 \rightarrow \bar{P}^+_{ijkl} = \tfrac{1}{2}(\bar{n}_i^{(1)}\bar{n}_k^{(1)}\delta_{jl} + \delta_{ik}\bar{n}_j^{(1)}\bar{n}_l^{(1)}) \quad (46a)$$

$$(b)\ \varepsilon^{(1)} > 0, \varepsilon^{(2)} > 0, \varepsilon^{(3)} \leqslant 0 \rightarrow \bar{P}^+_{ijkl} = \tfrac{1}{2}(\bar{n}_i^{(1)}\bar{n}_k^{(1)}\delta_{jl} + \delta_{ik}\bar{n}_j^{(1)}\bar{n}_l^{(1)})$$

$$+\tfrac{1}{2}(\bar{n}_i^{(2)}\bar{n}_k^{(2)}\delta_{jl} + \delta_{ik}\bar{n}_j^{(2)}\bar{n}_l^{(2)}) \quad (46b)$$

$$(c)\ \varepsilon^{(1)} > 0, \varepsilon^{(2)} > 0, \varepsilon^{(3)} > 0 \rightarrow \bar{P}^+_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (46c)$$

Although it may not seem obvious, eqns (46a, b) exhibit all the desirable features mentioned in previous sections: major and minor symmetries, same structure of positive and negative operators with same number of eigenvalues, and equivalence to the fourth-order symmetric identity tensor in the case of three eigenvalues of the same sign. Verification of these properties is a simple exercise by using relation (41).

Two cases are again considered with one or two positive principal strains. In the first case, differentiation of eqn (46a) leads to

$$\dot{P}^{+}_{rskl} = \tfrac{1}{2}(\dot{\bar{n}}^{(1)}_{r}\bar{n}^{(1)}_{k} + \bar{n}^{(1)}_{r}\dot{\bar{n}}^{(1)}_{k})\delta_{sl} + \tfrac{1}{2}\delta_{rk}(\dot{\bar{n}}^{(1)}_{s}\bar{n}^{(1)}_{l} + \bar{n}^{(1)}_{s}\dot{\bar{n}}^{(1)}_{l}). \tag{47}$$

This and eqn (46a) can be substituted in eqn (25) to obtain $\dot{d}^{sp}$ for unspecified (fully symmetric) stiffness degradation ΔE_{pqrs} as

$$\dot{d}^{sp} = \Delta E_{pqrs}(\varepsilon^{(1)^2}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(1)}_{r}\dot{\bar{n}}^{(1)}_{s} + \varepsilon^{(1)}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(1)}_{r}\varepsilon_{sk}\dot{\bar{n}}^{(1)}_{k}). \tag{48}$$

If two principal strains are positive, differentiation of eqn (46b) yields

$$\dot{P}^{+}_{rskl} = \tfrac{1}{2}(\dot{\bar{n}}^{(1)}_{r}\bar{n}^{(1)}_{k} + \bar{n}^{(1)}_{r}\dot{\bar{n}}^{(1)}_{k})\delta_{sl} + \tfrac{1}{2}\delta_{rk}(\dot{\bar{n}}^{(1)}_{s}\bar{n}^{(1)}_{l} + \bar{n}^{(1)}_{s}\dot{\bar{n}}^{(1)}_{l})$$
$$+ \tfrac{1}{2}(\dot{\bar{n}}^{(2)}_{r}\bar{n}^{(2)}_{k} + \bar{n}^{(2)}_{r}\dot{\bar{n}}^{(2)}_{k})\delta_{sl} + \tfrac{1}{2}\delta_{rk}(\dot{\bar{n}}^{(2)}_{s}\bar{n}^{(2)}_{l} + \bar{n}^{(2)}_{s}\dot{\bar{n}}^{(2)}_{l}). \tag{49}$$

This and eqn (46b) can be substituted in eqn (25) to obtain $\dot{d}^{sp}$ for unspecified (fully symmetric) stiffness degradation as

$$\dot{d}^{sp} = \Delta E_{pqrs}(\varepsilon^{(1)^2}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(1)}_{r}\dot{\bar{n}}^{(1)}_{s} + \varepsilon^{(2)^2}\bar{n}^{(2)}_{p}\bar{n}^{(2)}_{q}\bar{n}^{(2)}_{r}\dot{\bar{n}}^{(2)}_{s} + \varepsilon^{(1)}\varepsilon^{(2)}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(2)}_{r}\dot{\bar{n}}^{(2)}_{s}$$
$$+ \varepsilon^{(1)}\varepsilon^{(2)}\bar{n}^{(2)}_{p}\bar{n}^{(2)}_{q}\bar{n}^{(1)}_{r}\dot{\bar{n}}^{(1)}_{s} + \varepsilon^{(1)}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(1)}_{r}\varepsilon_{sk}\dot{\bar{n}}^{(1)}_{k} + \varepsilon^{(2)}\bar{n}^{(2)}_{p}\bar{n}^{(2)}_{q}\bar{n}^{(2)}_{r}\varepsilon_{sk}\dot{\bar{n}}^{(2)}_{k}$$
$$+ \varepsilon^{(1)}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(2)}_{r}\varepsilon_{sk}\dot{\bar{n}}^{(2)}_{k} + \varepsilon^{(2)}\bar{n}^{(2)}_{p}\bar{n}^{(2)}_{q}\bar{n}^{(1)}_{r}\varepsilon_{sk}\dot{\bar{n}}^{(1)}_{k}). \tag{50}$$

3.6.1. *Isotropic degradation.* Expressions (48) and (50) for the spurious dissipation rate $\dot{d}^{sp}$ contain the same type of contributions as the previous examples of the same type, eqn (36), plus new ones of the form $\Delta E_{pqrs}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(1)}_{r}\varepsilon_{sk}\dot{\bar{n}}^{(1)}_{k}$ and $\Delta E_{pqrs}\bar{n}^{(1)}_{p}\bar{n}^{(1)}_{q}\bar{n}^{(2)}_{r}\varepsilon_{sk}\dot{\bar{n}}^{(2)}_{k}$. The old ones were already discussed and vanish when ΔE_{pqrs} is isotropic, in Section 3.4. The new ones also vanish for isotropic degradation, which can be easily verified by substituting eqn (36), developing the products and using $\bar{n}^{(\alpha)}_{i}\bar{n}^{(\beta)}_{i} = \delta_{\alpha\beta}$ and $\varepsilon_{ij}\bar{n}^{(1)}_{i}\dot{\bar{n}}^{(1)}_{j} = 0$. Consequently, $\dot{d}^{sp} = 0$ for isotropic degradation also with the new projection operators.

3.6.2. *Anisotropic degradation.* By comparison with Section 3.4.2, the spurious dissipation rate for anisotropic degradation contains the same non-zero terms plus some new ones that do not vanish either. Therefore, spurious dissipation may also take place with the new projection operator proposed and anisotropic degradation.

4. E$^+$/E$^-$-BASED FORMULATIONS

4.1. *General*

This section examines a different class of procedures for MCR effects, which consist of decomposing stresses (or strains) into positive and negative parts according to eqns (18a, b) (or 19,a b), and using different compliance or stiffness tensors for each part. This may be expressed as:

$$\varepsilon_{ij} = C^{+}_{ijkl}\sigma^{+}_{kl} + C^{-}_{ijkl}\sigma^{-}_{kl} \quad \text{or} \quad \sigma_{ij} = E^{+}_{ijkl}\varepsilon^{+}_{kl} + E^{-}_{ijkl}\varepsilon^{-}_{kl}. \tag{51a, b}$$

Although not presented in this way in the literature (Mazars and Pijaudier-Cabot, 1989), for the sake of uniformity in the terminology and for ease of comparison, we substitute $\boldsymbol{\sigma}^+$ and $\boldsymbol{\sigma}^-$ (or $\boldsymbol{\varepsilon}^+$ and $\boldsymbol{\varepsilon}^-$) in terms of $\boldsymbol{\sigma}$ (or $\boldsymbol{\varepsilon}$) and the projection operators (20a–d), and obtain

$$\varepsilon_{ij} = C^{ac}_{ijkl}\sigma_{kl}; \quad C^{ac}_{ijkl} = C^{+}_{ijpq}P^{+}_{pqkl} + C^{-}_{ijpq}P^{-}_{pqkl} \tag{52a, b}$$

or their dual expressions

$$\sigma_{ij} = E^{ac}_{ijkl}\varepsilon_{kl}; \quad E^{ac}_{ijkl} = E^{+}_{ijpq}\bar{P}^{+}_{pqkl} + E^{-}_{ijpq}\bar{P}^{-}_{pqkl}. \tag{53a, b}$$

These equations can be further expanded assuming

$$C^{+}_{ijkl} = C^{0}_{ijkl} + \Delta C^{+}_{ijkl} \quad \text{and} \quad C^{-}_{ijkl} = C^{0}_{ijkl} + \Delta C^{-}_{ijkl} \tag{54a, b}$$

or, for the dual formulation

$$E^{+}_{ijkl} = E^{0}_{ijkl} - \Delta E^{+}_{ijkl} \quad \text{and} \quad E^{-}_{ijkl} = E^{0}_{ijkl} - \Delta E^{-}_{ijkl}. \tag{55a, b}$$

By assuming $\mathbf{P}^{+} + \mathbf{P}^{-} = \mathbf{I}^{4}$ and $\bar{\mathbf{P}}^{+} + \bar{\mathbf{P}}^{-} = \mathbf{I}^{4}$ (this condition is satisfied by all definitions of the projection operators considered in Section 3), one obtains equations analogous to eqns (21a) or (22a), with

$$\Delta C^{ac}_{ijkl} = \Delta C^{+}_{ijkl}P^{+}_{ijkl} + \Delta C^{-}_{ijkl}P^{-}_{ijkl} \quad \text{or} \quad \Delta E^{ac}_{ijkl} = \Delta E^{+}_{ijkl}\bar{P}^{+}_{ijkl} + \Delta E^{-}_{ijkl}\bar{P}^{-}_{ijkl}. \tag{56a, b}$$

Note that these expressions differ from eqn (21a, b) in that the operator only acts on one side of the positive and negative increments of intrinsic compliance (or stiffness) and major symmetry is no longer guaranteed for $\Delta\mathbf{C}^{ac}$ (or $\Delta\mathbf{E}^{ac}$), and therefore neither for $\mathbf{C}^{ac}$ (or $\mathbf{E}^{ac}$). The lack of symmetry, however, is not a problem by itself, but the consequences normally associated with it, such as the lack of a well-defined energy potential and dissipation or generation upon closed-loop loading. These associated concepts are however strictly valid only if the (non-symmetric) stiffness or compliance remain constant during the closed loop. This is not necessarily the case here since the active compliance depends on the projection operators, and these may change during a closed cycle causing zero energy dissipation even with a non-symmetric (but non-constant) active compliance. To address these important points, the same methodology used in the previous section is applied to the strain-based formulation with only positive decrement of stiffness (i.e. $\mathbf{E}^{-} = \mathbf{E}^{0}$ which means $\Delta\mathbf{E}^{-} = 0$). Also, the notation $\Delta\mathbf{E}^{+} = \Delta\mathbf{E}$ is adopted for simplicity, with the resulting expression

$$E^{ac}_{ijkl} = E^{0}_{ijkl} - \Delta E_{ijpq}\bar{P}^{+}_{pqkl}. \tag{57}$$

The conclusions obtained in this case may be readily generalized to other $\mathbf{E}^{+}/\mathbf{E}^{-}$ formulations.

4.2. *Spurious dissipation*

Differentiation of eqn (57) for constant $\Delta\mathbf{E}$ yields

$$\dot{E}^{ac}_{ijkl} = -\Delta E_{ijpq}\dot{\bar{P}}^{+}_{pqkl}. \tag{58}$$

Introduction into eqn (16a) leads to the basic expression of the spurious dissipation rate:

$$\dot{d}^{sp} = \tfrac{1}{2}\varepsilon_{pq}\Delta E_{pqrs}\dot{\bar{P}}^{+}_{rskl}\varepsilon_{kl}. \tag{59}$$

To proceed further, one has to assume specific expressions for the projection operator and the stiffness degradation. First, the projection operators in the previous section are revisited and the corresponding $\dot{d}^{sp}$ obtained. For Ortiz's and Simo and Ju's projection operators with one direction of positive principal strain, eqns (32a) and (40a), $\bar{P}^{+}$ in eqn (33) is introduced into eqn (59), yielding

$$\dot{d}^{\text{sp}} = \varepsilon_{pq}\Delta E_{pqrs}\varepsilon^{(1)}\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)}. \tag{60}$$

For two directions of positive principal strain with the original proposal by Ortiz (32b), substitution of $\dot{P}^+$ from eqn (35) into eqn (59) yields

$$\dot{d}^{\text{sp}} = \varepsilon_{pq}\Delta E_{pqrs}(\varepsilon^{(1)}\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)} + \varepsilon^{(2)}\bar{n}_r^{(2)}\dot{\bar{n}}_s^{(2)}). \tag{61}$$

For two directions of positive principal strain with Simo and Ju's operator (40b), substitution of $\dot{P}^+$ from eqn (43) into eqn (59) yields [the symmetrized version, eqn (42), leads to the same result]

$$\dot{d}^{\text{sp}} = \varepsilon_{pq}\Delta E_{pqrs}(\varepsilon^{(1)}\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)} + \varepsilon^{(2)}\bar{n}_r^{(2)}\dot{\bar{n}}_s^{(2)} + \varepsilon_{ij}(\bar{n}_i^{(1)}\dot{\bar{n}}_j^{(2)} + \dot{\bar{n}}_i^{(1)}\bar{n}_j^{(2)})\bar{n}_r^{(1)}\bar{n}_s^{(2)}). \tag{62}$$

For one single direction of positive principal strain and the new projection operator proposed in Section 3.6 (46a), substitution of $\dot{P}^+$ from eqn (47) into eqn (59) yields

$$\dot{d}^{\text{sp}} = \tfrac{1}{2}\varepsilon_{pq}\Delta E_{pqrs}(\varepsilon^{(1)}\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)} + \bar{n}_r^{(1)}\varepsilon_{sk}\dot{\bar{n}}_k^{(1)}). \tag{63}$$

For two directions of positive principal strain and the new projection operator (46b), substitution of $\dot{P}^+$ from eqn (49) into eqn (59) yields

$$\dot{d}^{\text{sp}} = \tfrac{1}{2}\varepsilon_{pq}\Delta E_{pqrs}(\varepsilon^{(1)}\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)} + \bar{n}_r^{(1)}\varepsilon_{sk}\dot{\bar{n}}_k^{(1)} + \varepsilon^{(2)}\bar{n}_r^{(2)}\dot{\bar{n}}_s^{(2)} + \bar{n}_r^{(2)}\varepsilon_{sk}\dot{\bar{n}}_k^{(2)}). \tag{64}$$

4.2.1. *Isotropic degradation.* Substituting the general expression of isotropic degradation ΔE_{pqrs}, eqn (26), into each of the possible equations (60)–(64) we observe that all them lead to $\dot{d}^{\text{sp}} = 0$. To see that, consider first that the common factor on the left becomes $\varepsilon_{pq}\Delta E_{pqrs} = \Delta\lambda\varepsilon_{pp}\delta_{rs} + 2\Delta\mu\varepsilon_{rs}$, and this is multiplied by only three types of factors: $\bar{n}_r^{(1)}\dot{\bar{n}}_s^{(1)}$, $\bar{n}_r^{(1)}\bar{n}_s^{(2)}$ and $\bar{n}_r^{(1)}\varepsilon_{sk}\dot{\bar{n}}_k^{(1)}$, all of which can be easily found to be zero by using that $\bar{n}_i^{(1)}\bar{n}_i^{(2)} = 0$, $\varepsilon_{ij}\bar{n}_i^{(1)} = \varepsilon^{(1)}\bar{n}_i^{(1)}$, $\varepsilon_{ij}\bar{n}_i^{(1)}\dot{\bar{n}}_j^{(1)} = 0$ and $\varepsilon_{ij}\bar{n}_i^{(1)}\bar{n}_j^{(2)} = 0$.

4.2.2. *Anisotropic degradation.* For anisotropic degradation, $\dot{d}^{\text{sp}}$ is non-zero in general. Only one example of this will be considered, as in Section 3.4.2, with ΔE_{ijpq} given by eqn (28) and Ortiz–Simo–Ju's projection operator with one single positive principal strain value. For that case, one can introduce eqn (28) into eqn (60) to obtain

$$\dot{d}^{\text{sp}} = -\varepsilon^{(1)}(\varepsilon_{ij}d_id_j)\Delta E\dot{\theta}\cos\theta\sin\theta \tag{65}$$

where $\varepsilon_{ij}d_id_j$ is the normal strain in the direction of maximum degradation $\mathbf{d}$ (fixed in the material), and θ has the same meaning as in the previous section, i.e. the angle between $\bar{\mathbf{n}}^{(1)}$ and $\mathbf{d}$. Note that, similarly as in $\mathbf{P}^+/\mathbf{P}^-$ models, spurious dissipation occurs only when the axis of maximum principal strain rotates, i.e. $\dot{\theta} \neq 0$.

5. A SIMPLE EXAMPLE WITH SPURIOUS DISSIPATION–GENERATION

For definiteness, we consider the model of anisotropic degradation proposed in Section 3.3.2, with the intrinsic degradation of stiffness given by eqn (28). Let us assume that the MCR effects are taken into account with the $\mathbf{P}^+/\mathbf{P}^-$ procedure given in relation (23), with the projection operators for one single positive principal strain proposed by Ortiz eqn (32a), or Simo–Ju, eqn (40a) (same expression). Combining these equations, the stress–strain relation at a fixed state of degradation is

$$\sigma_{ij} = E^{ac}_{ijkl}\varepsilon_{kl}; \quad E^{ac}_{ijkl} = E^{0}_{ijkl} - \Delta E \cos^4 \theta \bar{n}^{(1)}_i \bar{n}^{(1)}_j \bar{n}^{(1)}_k \bar{n}^{(1)}_l \tag{66a,b}$$

where θ was defined in Section 3.3.2 as the angle between the direction $\mathbf{d}$ of maximum degradation (fixed material direction) and the direction of the positive principal strain $\bar{\mathbf{n}}^{(1)}$. For simplicity, it is assumed that $\mathbf{d}$ coincides with the x axis, and therefore θ is the angle between $\bar{\mathbf{n}}^{(1)}$ and the x axis. Note that with these assumptions, the reduction of active stiffness, eqns (66), has the same structure as the reduction of intrinsic stiffness, eqn (28), with $\Delta E \cos^4 \theta$ instead of ΔE, and $\bar{n}^{(1)}_i, \bar{n}^{(1)}_j, \ldots$, instead of $d_i d_j \ldots$. This means that, for a given direction of the positive principal strain $\bar{\mathbf{n}}^{(1)}$ that does not coincide with the direction of maximum anistropic degradation $\mathbf{d}$, the active stiffness tensor is equivalent to the intrinsic stiffness tensor (assuming all microcracks active) of a fictitious state of anisotropic degradation with maximum value $\Delta E \cos^4 \theta$ in the direction $\bar{\mathbf{n}}^{(1)}$. The two limiting cases are: when $\bar{\mathbf{n}}^{(1)} = \mathbf{d}$, with $\cos \theta = 1$ and the active stiffness equal to the intrinsic secant stiffness (all microcracks are active), and when $\bar{\mathbf{n}}^{(1)}$ is perpendicular to $\mathbf{d}$, with $\cos \theta = 0$ and the active stiffness equal to the initial one (all microcracks closed).

Let us consider now a closed excursion in strain space involving rotation of principal directions as follows:

(i) a certain positive value of normal strain $\varepsilon^{(1)}$ is applied in the direction of the y-axis with all other components of strain remaining zero;
(ii) the strain state is rotated 90° on the x–y plane so that the normal strain on the direction of the x-axis becomes $\varepsilon^{(1)}$ and all other components zero; and
(iii) the x-strain is decreased from $\varepsilon^{(1)}$ back to zero.

This strain history, depicted in Fig. 1, always contains one single principal strain positive, as assumed above. It can be argued whether in this case all three principal strains, and not only one, are $\geqslant 0$ and therefore strictly speaking $\mathbf{P}^+ = \mathbf{I}^4$ instead of (32a) and $\Delta E^{ac}_{ijkl} = E^0_{ijkl} - \Delta E d_i d_j d_k d_l$ instead of (66b). This argument, however, is readily resolved by considering that a sufficiently small negative value of the strain, instead of zero, is applied on the second and third axes, since then eqns (32a) and (66) hold necessarily and the same conclusions that follow are obtained.

Equations (66) yields the total value of stresses at each stage of the strain history described. Therefore it is straight forward to calculate the external work supplied to the material during each of the three load steps, and to obtain the net energy dissipation (or generation) during the closed loop

$$\Delta d^{sp} = \Delta w^{(i)} + \Delta w^{(ii)} + \Delta w^{(iii)}. \tag{67}$$

During the first step, $\bar{\mathbf{n}}^{(1)}$ coincides with the y-axis and $\cos \theta = 0$. According to eqns (66) this means that the reduction of stiffness is zero, and therefore the active stiffness is constant and equal to the initial stiffness $E^{ac}_{ijkl} = E^0_{ijkl} = \lambda^0 \delta_{ij}\delta_{kl} + \mu^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The external work supply during this first step is

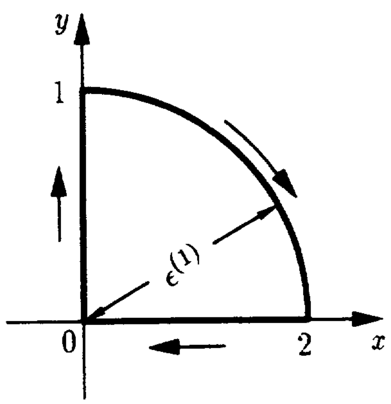

Fig. 1. Strain path.

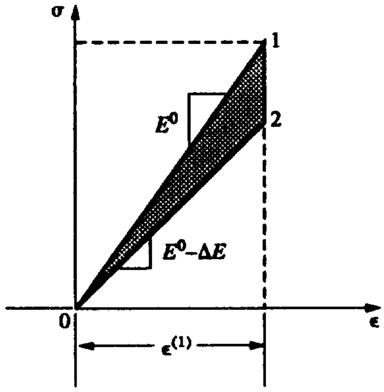

Fig. 2. Resulting stress–strain diagram.

$$\Delta w^{(i)} = \int_0^1 \sigma_{ij}\,d\varepsilon_{ij} = \int_{\varepsilon_{22}=0}^{\varepsilon_{22}=\varepsilon^{(1)}} E_{2222}\varepsilon_{22}\,d\varepsilon_{22} = \frac{1}{2}E^0\varepsilon^{(1)^2} \tag{68}$$

where $E^0 = \lambda^0 + 2\mu^0$. This amount of work corresponds to the area enclosed below the line 0–1 with slope E^0 in Fig. 2. In the third load step (the second one is considered later), $\bar{\mathbf{n}}^{(1)}$ coincides with the x-axis, which means that $\cos\theta = 1$ and the reduction of the active stiffness is maximum. The external work supply during this step is:

$$\Delta w^{(iii)} = \int_0^1 \sigma_{ij}\,d\varepsilon_{ij} = \int_{\varepsilon_{11}=\varepsilon^{(1)}}^{\varepsilon_{11}=0} E_{1111}\varepsilon_{11}\,d\varepsilon_{11} = -\frac{1}{2}(E^0 - \Delta E)\varepsilon^{(1)^2}. \tag{69}$$

The absolute value of this quantity corresponds to the area enclosed below line 2–0 with slope $E^0 - \Delta E$ in Fig. 2.

For the remaining second step, one can write the equations for strain [using eqn (17b)], its rate of variation (by differentiation, taking into account that $\varepsilon^{(1)}$ is constant and only $\bar{\mathbf{n}}^{(1)}$ changes), and the equations for stresses [from eqns (66), using that $\bar{n}_i^{(1)}\bar{n}_i^{(1)} = 1$], as

$$\varepsilon_{ij} = \varepsilon^{(1)}\bar{n}_i^{(1)}\bar{n}_j^{(1)} \tag{70}$$

$$\dot{\varepsilon}_{ij} = \varepsilon^{(1)}(\dot{\bar{n}}_i^{(1)}\bar{n}_j^{(1)} + \bar{n}_i^{(1)}\dot{\bar{n}}_j^{(1)}) \tag{71}$$

$$\sigma_{ij} = \lambda^0\varepsilon^{(1)}\delta_{ij} + (2\mu^0 - \Delta E\cos^4\theta)\varepsilon^{(1)}\bar{n}_i^{(1)}\bar{n}_j^{(1)}. \tag{72}$$

With eqns (71) and (72) and taking into account that $\bar{n}_i^{(1)}\dot{\bar{n}}_i^{(1)} = 0$, the rate of external work supply is

$$\dot{w}^{(ii)} = \sigma_{ij}\dot{\varepsilon}_{ij} = \varepsilon^{(1)^2}(\lambda^0\delta_{ij} + (2\mu^0 - \Delta E\cos^4\theta)\bar{n}_i^{(1)}\bar{n}_j^{(1)})(\dot{\bar{n}}_i^{(1)}\bar{n}_j^{(1)} + \bar{n}_i^{(1)}\dot{\bar{n}}_j^{(1)}) = 0. \tag{73}$$

This means that $\Delta w^{(ii)} = 0$, and from eqn (67), the total dissipation is

$$\Delta d^{sp} = \tfrac{1}{2}E^0\varepsilon^{(1)^2} - \tfrac{1}{2}(E^0 - \Delta E)\varepsilon^{(1)^2} = \tfrac{1}{2}\Delta E\varepsilon^{(1)^2}, \tag{74}$$

i.e. during the closed cycle considered, net energy dissipation takes place (!) and its amount corresponds to the shaded area enclosed by the triangle 0–1–2 in Fig. 2. Also, it must be noted that the same amount of energy can be generated instead of dissipated if the reverse path of the same load history is considered (!).

Further considerations are made in order to gain insight in the results just obtained. Equation (73) indicates that, with the type of active stiffness, eqns (66), a pure rotation of principal strains does not require any external work, even if the stresses are changing during the rotation process. The normal stress $\bar{\sigma}$ in the direction of the principal strain at every stage of the rotation process is calculated from eqn (72) as $\bar{\sigma} = \sigma_{ij}\bar{n}_i^{(1)}\bar{n}_j^{(1)} = (E^0 - \Delta E\cos^4\theta)\varepsilon^{(1)}$, with E^0 defined above. In this expression $\varepsilon^{(1)}$ remains constant, but not the angle θ, that changes from 90 to 0°, causing a reduction of $\bar{\sigma}$ from $E^0\varepsilon^{(1)}$ to $(E^0 - \Delta E)\varepsilon^{(1)}$. This is consistent with the results represented in Fig. 2 since, at the beginning of the second step, $\bar{\sigma}$ must coincide with the σ_{22} obtained at the end of the first step and, at the end of the second step, $\bar{\sigma}$ should coincide with the σ_{11} calculated at the beginning of the third step. The change of $\bar{\sigma}$ is therefore correct, and since the corresponding strain $\varepsilon^{(1)}$ remains constant, the fact that this change in $\bar{\sigma}$ does not require external work by itself is not surprising. But on the other hand, the difference in energy between the loading during the first part of the cycle (with initial stiffness) and the unloading (with reduced stiffness), indicates that a consistent model with a well-defined energy potential requires external work during the rotation process, with an amount equal to that difference (shaded area in Fig. 2) that now appears as spurious dissipation. Since variation of normal stresses on the plane perpendicular to $\bar{\mathbf{n}}^{(1)}$ does not require work, the only way to obtain that work should come from some shear stresses on the same plane. This is confirmed if one evaluates the components of the strain rate, eqn (71), with $\bar{\mathbf{n}}^{(1)} = (\cos\theta,\ \sin\theta,\ 0)'$ and $\dot{\bar{\mathbf{n}}}^{(1)} = (-\dot{\theta}\sin\theta, \dot{\theta}\cos\theta, 0)'$, for $\theta = 0$ (which would be the expression of $\dot{\varepsilon}$ on the rotated axes at any time during the second step), which are all zeros except for the component $\dot{\varepsilon}_{12} = \dot{\theta}$. Obtaining shear stresses on the plane of principal strain would require non-zero normal–shear cross terms in the active stiffness tensor expressed in a rotated reference system. This, however, is not possible with eqns (66) and other active stiffness tensors generated in a similar way, because $\bar{\mathbf{n}}^{(1)}$ is always one of the directions of orthotropy of the tensor itself, and that implies necessarily zero values for that kind of cross terms.

To conclude the example, Δd^{sp} can also be obtained from direct integration of the expression in Section 3.4.2. for the rate of spurious dissipation, eqn (38). From that formula with $\varepsilon^{(1)} =$ constant, the dissipation is

$$\Delta d^{sp} = \int_{\theta=\pi/4}^{\theta=0} -2\Delta E\varepsilon^{(1)^2}\cos^3\theta\sin\theta\,\mathrm{d}\theta = -2\Delta E\varepsilon^{(1)^2}\left(-\frac{\cos^4[0]}{4} + \frac{\cos^4[\pi/4]}{4}\right) = \tfrac{1}{2}\Delta E\varepsilon^{(1)^2} \tag{75}$$

which coincides with eqn (74) as should be expected.

6. CONCLUSIONS AND FINAL REMARKS

(1) Most representations of microcrack closure in stiffness degradation models are based either on the definition of positive–negative projection operators, or on the use of different stiffnesses in tension or compression. The implementation of such procedures for a fixed state of degradation should follow the hyperelastic postulate of a well-defined energy potential which assumes that no dissipation takes place upon closed-cycle excursions in stress or stress space. This particular aspect has apparently not been considered in most of the recovery models proposed in the literature.

(2) The concept of spurious dissipation rate, that should be always zero for energy-consistent formulations, was developed with considerable generality. An explicit expression was obtained and specified for the most familiar projection operators combined with two types of stiffness degradation: general isotropic degradation, and a particular type of anisotropic degradation.

(3) All definitions found in the literature for the positive and negative projection operators exhibit shortcomings such as lack of minor symmetries, dissimilar structures for the equivalent states of positive and negative stress or strain, non-equivalence of the

operator for three principal values of the same sign with the identity tensor, etc. A new definition of the projection operators was proposed that overcomes most of these problems. Nevertheless, the results obtained with regard to spurious dissipation are not influenced by the choice of projection operators.

(4) Detailed inspection of the rate of spurious dissipation for different recovery formulations ($\mathbf{P}^+/\mathbf{P}^-$, $\mathbf{E}^+/\mathbf{E}^-$, different projection operators), reveals that all of them turn out to be energy-consistent when used in the context of isotropic degradation, but all of them fail when anisotropic degradation is considered. This is illustrated with a simple example involving a popular recovery model and a 90° rotation of a single tensile principal strain.

(5) The preceding conclusion appears to be of general validity for all MCR procedures based on positive/negative decomposition of stress or strain. The only type of constitutive formulation that has been suggested to be capable of representing MCR effects with anisotropic degradation in an energy-conservative manner (Carol and Willam, 1994) is the microplane model. In that formulation, the material laws and the recovery effects due to microcrack closure are treated individually for each plane with different orientation, while stress, stiffnesses and other tensorial quantities are obtained *a posteriori* by integration of microplane contributions over the hemisphere.

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